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Pointwise Estimates of the Hermitian Interpolation

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We obtain a pointwise estimate of the deviation of a function from her Hermitian interpolating polynomial.

1

The well-known Whitney Theorem [1] allows one to evaluate the largest deviation of a continuous function f from the algebraic polynomial p of degree $\leqslant n-1$ which coincides with f at n equidistant points $x_1 = a < x_2 < \cdots < x_n = b$. Namely,

$$\max_{a \leqslant x \leqslant b} |f(x) - p(x)| \leqslant C\omega_n\left(f; \frac{b-a}{n}\right), \tag{A}$$

where $\omega_n(f; \cdot)$ is the nth-order modulus of smoothness for f.

It is clear that the value of $|f(x) - p(x)|$ is essentially smaller than the largest deviation when x is close to a node of interpolation. Such estimates of the deviation taking into consideration the position of x (pointwise estimates) were earlier obtained in the particular case of two nodes, they are both the endpoints of the interval, having the same multiplicity r.

First, for $r = 1$ A. F. Timan and L. I. Strukov in [2] proved that for any $f \in C[-1; 1]$ and $x \in [-1; 1]$

$$|f(x) - p(x)| \leqslant 15\omega_2(f; (1-x^2)^{1/2}). \tag{B}$$

Then, for any natural r we proved in [3] that for any $f \in C^{r-1}[-1; 1]$ and $x \in [-1; 1]$

$$|f(x) - p(x)| \leqslant C_r(1-x^2)^{r-1}\,\omega_{r+1}\left(f^{(r-1)}; \frac{2}{r+1}(1-x^2)^{1/(r+1)}\right). \tag{C}$$

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In this paper we investigate the error of Hermitian interpolation with any nodes and with arbitrary distribution of multiplicities in the nodes. The pointwise estimate (D) obtained here is a direct generalization of the inequality (C), this estimate also gives some refinement to the Whitney result (A).

In the next section we introduce necessary notations and formulate the main result of the paper.

2

In what follows a and b $(a<b)$ are given real numbers; $\|f\|$ means $\|f\|_{C[a,b]}$; $\mathscr{P}_k$ is a linear space of algebraic polynomials of degree $\leqslant k$; $\omega_k(f;h)=\sup_{0\leqslant t\leqslant h}\{|\Delta_t^k f(x)|;x, x+kh\in[a,b]\}$, where $\Delta_t^k f(x)=\sum_{\nu=0}^k(-1)^\nu\binom{k}{\nu}f(x+\nu t)$.

Let $x_1, x_2, ..., x_n$ be a set of nodes such that $a\leqslant x_1\leqslant x_2\leqslant\cdots\leqslant x_n\leqslant b$ and let $r_1, r_2, ..., r_n\in\mathbb{N}$ be the corresponding multiplicities. We denote $r=r_1+r_2+\cdots+r_n$, $\hat{r}=\max\{r_1, r_2, ..., r_n\}$, $r_i'=\min\{r_i, \hat{r}-1\}$; $A_1(x)=\prod_{i=1}^n(x-x_i)^{r_i'}$, $\alpha(x)=\prod_{i:\, r_i=\hat{r}}(x-x_i)$, $\alpha=\max_{a\leqslant x\leqslant b}|\alpha(x)|$ and finally $A(x)=A_1(x)\,\alpha(x)$.

We will formulate the main result making use of these notations.

THEOREM. *Let* $f\in C^{r-1}[a,b]$ *and* $p\in\mathscr{P}_{r-1}$ *be such that* $p^{(\nu)}(x_i)=f^{(\nu)}(x_i)$, $\nu=0,1,2,...,r_i-1$, $i=1,2,...,n$.

Then there exists a constant $C>0$ *depending only on the position of the nodes in* $[a,b]$ *and on their multiplicities, such that for every* $x\in[a,b]$ *the following inequality holds*

$$|f(x)-p(x)|\leqslant C\,|A_1(x)|\,\omega_{r-\hat{r}+1}\left(f^{(\hat{r}-1)};\frac{b-a}{r-\hat{r}+1}\left(\frac{\alpha(x)}{\alpha}\right)^{1/(r-\hat{r}+1)}\right). \quad \text{(D)}$$

In Sections 3–6 we state some lemmas used in the proof of the main theorem. In the sequel it will be convenient to use the following notation

$$[x,p]=\begin{cases}1, & p=0\\ x(x+1)\cdots(x+p-1), & p=1,2,3,....\end{cases}$$

3

In the next two lemmas we put $m\in\mathbb{N}$, $x_1\leqslant\cdots\leqslant x_m$, $k_1, ..., k_m\in\mathbb{N}$, $k=k_1+k_2+\cdots+k_m$, $\Phi(x)=\prod_{i=1}^m(x-x_i)^{k_i}$.

LEMMA 1. *Let $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Than the inequalities are valid*

$$\left|\frac{d^p}{dx^p}\frac{1}{\Phi(x)}\right| \leqslant \frac{[k, p]}{(x-x_m)^{k+p}} \qquad \text{if} \quad x > x_m$$

and

$$\left|\frac{d^p}{dx^p}\frac{1}{\Phi(x)}\right| \leqslant \frac{[k, p]}{(x_1-x)^{k+p}} \qquad \text{if} \quad x < x_1.$$

Proof. By Leibniz formula

$$\frac{d^p}{dx^p}\prod_{i=1}^{m}\frac{1}{(x-x_i)^{k_i}} = \sum_{p_1+\cdots+p_m=p}\frac{p!}{p_1!\cdots p_m!}\prod_{i=1}^{m}(-1)^{p_i}\frac{[k_i, p_i]}{(x-x_i)^{k_i+p_i}}$$
$$= (-1)^p p! \sum_{p_1+\cdots+p_m=p}\prod_{i=1}^{m}\frac{1}{p_i!}\frac{[k_i, p_i]}{(x-x_i)^{k_i+p_i}}.$$

In particular, if $x_1 = x_2 = \cdots = x_m = a$ then

$$\frac{d^p}{dx^p}\frac{1}{(x-a)^k} = (-1)^p p! \sum_{p_1+\cdots+p_m=p}\prod_{i=1}^{m}\frac{1}{p_i!}\frac{[k_i, p_i]}{(x-a)^{k_i+p_i}}.$$

Therefore, for $x > x_m$

$$\left|\frac{d^p}{dx^p}\frac{1}{\Phi(x)}\right| \leqslant \left|\frac{d^p}{dx^p}\frac{1}{(x-x_m)^k}\right| = \frac{[k, p]}{(x-x_m)^{k+p}}$$

and for $x < x_1$

$$\left|\frac{d^p}{dx^p}\frac{1}{\Phi(x)}\right| \leqslant \left|\frac{d^p}{dx^p}\frac{1}{(x_1-x)^k}\right| = \frac{[k, p]}{(x_1-x)^{k+p}}.$$

LEMMA 2. *Let $p, q \in \mathbb{N}_0$, $q \leqslant k+p$, $\varepsilon > 0$,*

$$H_1 = \{(x, s) \mid x \geqslant x_m + \varepsilon,\ x_m \leqslant s \leqslant x_m + 2\varepsilon\}$$

and

$$H_2 = \{(x, s) \mid x \leqslant x_1 - \varepsilon,\ x_1 - 2\varepsilon \leqslant s \leqslant x_1\}.$$

Then for $(x, s) \in H = H_1 \cup H_2$ the following inequality holds

$$\left|\frac{d^p}{dx^p}\frac{(x-s)^q}{\Phi(x)}\right| \leqslant \frac{1}{\varepsilon^{k+p-q}}\sum_{i=0}^{\min(p, q)}\binom{p}{i}[q-i+1, i][k, p-i].$$

Proof. If $(x, s) \in H_1$ then $|x-s| \leqslant x - x_m$. Hence according to Lemma 1

$$\left|\frac{d^p}{dx^p}\frac{(x-s)^q}{\Phi(x)}\right| \leqslant \sum_{i=0}^{\min(p,q)} \binom{p}{i} [q-i+1, i][k, p-i] \frac{|x-s|^{q-i}}{(x-x_m)^{k+p-i}}$$

$$\leqslant \frac{1}{(x-x_m)^{k+p-q}} \sum_{i=0}^{\min(p,q)} \binom{p}{i} [q-i+1, i][k, p-i].$$

It remains to note that $x - x_m \geqslant \varepsilon$.

The case $(x, s) \in H_2$ can be considered in the same way.

4

Let $a \leqslant x_1 < x_2 < \cdots < x_n \leqslant b$; $r_1, r_2, ..., r_n \in \mathbb{N}$, $r = \sum_{i=1}^n r_i$, $A(x) = \prod_{i=1}^n (x-x_i)^{r_i}$. We denote by $G(x, s)$ Green's function for the generalized boundary value problem:

$$y^{(r)}(x) = A(x); \qquad y^{(\nu)}(x_i) = 0, \qquad \nu = 0, 1, ..., r_i - 1, \quad i = 1, 2, ..., n.$$

LEMMA 3. *Let* $\mu \in \{0, 1, ..., r-1\}$, $j \in \{1, ..., n\}$. *We denote*

$$E_j^1 = \left\{(x, s) \,\middle|\, x_j \leqslant x \leqslant \frac{x_j + x_{j+1}}{2}, x_j \leqslant s \leqslant x\right\},$$

$$E_j^2 = \left\{(x, s) \,\middle|\, \frac{x_j + x_{j+1}}{2} \leqslant x \leqslant x_{j+1}, x \leqslant s \leqslant x_{j+1}\right\}, \qquad E_j = E_j^1 \cup E_j^2.$$

Then there is a constant $D_{j,\mu}$ *such that the following inequality*

$$\left|\frac{\partial^\mu}{\partial s^\mu} G(x, s)\right| \leqslant D_{j,\mu} |A(x)| + \chi_{E_j}(x, s) \frac{|x-s|^{r-\mu-1}}{(r-\mu-1)!}$$

holds for all $s \in (x_j, x_{j+1})$, $x \in [a, b]$ (χ_E *denotes the indicator of the set* E).

Proof. For every fixed $s \in (x_j, x_{j+1})$

$$G(x, s) = \begin{cases} g_{1,s}(x) & \text{for} \quad a \leqslant x \leqslant s \leqslant b, \\ g_{2,s}(x) & \text{for} \quad a \leqslant s \leqslant x \leqslant b, \end{cases} \tag{1}$$

where $g_{1,s}, g_{2,s} \in \mathcal{P}_{r-1}$, $g_{1,s}$ is divisible by the polynomial $A_j^-(x) = \prod_{i=1}^j (x-x_i)^{r_i}$, $g_{2,s}(x)$ is divisible by $A_j^+(x) = \prod_{i=j+1}^n (x-x_i)^{r_i}$ and $g_{2,s}(x) - g_{1,s}(x) = (x-s)^{r-1}/(r-1)!$.

It is clear that $p_{1,s} = g_{1,s}/A_j^- \in \mathcal{P}_{r_j^+ - 1}$, $p_{2,s} = g_{2,s}/A_j^+ \in \mathcal{P}_{r_j^- - 1}$, where $r_j^- = r_1 + \cdots + r_j$ and $r_j^+ = r_{j+1} + \cdots + r_n$.

The polynomials $p_{1,s}$ and $p_{2,s}$ are uniquely determined by the condition

$$A_j^+(x)\,p_{2,s}(x)-A_j^-(x)\,p_{1,s}(x)=\frac{(x-s)^{r-1}}{(r-1)!} \tag{2}$$

since A_j^+ and A_j^- are relatively prime. The equality (2) can be represented as a system of linear equations in unknowns $\alpha_i(s)$, $\beta_i(s)$, the coefficients of the polynomials $p_{1,s}$, $p_{2,s}$. Solving this system we find that $\alpha_i(s)$, $\beta_i(s)$ are algebraic polynomials of degree $<r-1$ and hence $p_{1,s}$ and $p_{2,s}$ are algebraic polynomials of two variables.

By differentiating both parts of the relation (2) we obtain

$$A_j^+(x)\frac{\partial^\mu}{\partial s^\mu}p_{2,s}(x)-A_j^-(x)\frac{\partial^\mu}{\partial s^\mu}p_{1,s}(x)=\frac{(-1)^\mu(x-s)^{r-\mu-1}}{(r-\mu-1)!}. \tag{3}$$

Let $x\geqslant(x_j+x_{j+1})/2$. From (3) we find for functions

$$z_s(x)=\frac{A_j^+(x)}{A_j^-(x)}\frac{\partial^\mu}{\partial s^\mu}p_{2,s}(x),\qquad \varphi_s(x)=\frac{1}{A_j^-(x)}\frac{(-1)^\mu(x-s)^{r-\mu-1}}{(r-\mu-1)!}$$

that

$$z_s(x)=\frac{\partial^\mu}{\partial s^\mu}p_{1,s}(x)+\varphi_s(x).$$

Hence

$$\frac{\partial^{r_j^+}}{\partial x^{r_j^+}}z_s(x)=\frac{\partial^{r_j^+}}{\partial x^{r_j^+}}\varphi_s(x).$$

Moreover, $(\partial^\nu/\partial x^\nu)\,z_s(x_i)=0$ for $\nu\leqslant r_i-1$, $i=j+1,\ldots,n$. This means that $z_s(x)$ is equal to its deviation of the function $\varphi_s(x)$ from the Hermitian interpolating polynomial with r_i-fold nodes at the points x_i, $i=j+1,\ldots,n$.

From the properties of divided differences it follows that

$$\left|\frac{1}{A_j^-(x)}\frac{\partial^\mu}{\partial s^\mu}g_{2,s}(x)\right|=|z_s(x)|\leqslant\frac{1}{(r_j^+)!}\,\|\varphi_s^{(r_j^+)}\|_{C[(x_j+x_{j+1})/2;\,b]}\,|A_j^+(x)|.$$

Therefore, for every $x\geqslant(x_j+x_{j+1})/2$ and $s\in[x_j,x_{j+1}]$ we have

$$\left|\frac{\partial^\mu}{\partial s^\mu}g_{2,s}(x)\right|\leqslant\frac{1}{(r_j^+)!}\frac{1}{(r-\mu-1)!}\max_{\substack{x_j\leqslant s\leqslant x_{j+1}\\ x\geqslant(x_j+x_{j+1})/2}}\left|\frac{\partial^{r_j^+}}{\partial x^{r_j^+}}\frac{(x-s)^{r-\mu-1}}{A_j^-(x)}\right|\,|A(x)|.$$

In view of Lemma 2 we obtain that for such x and s

$$\left|\frac{\partial^\mu}{\partial s^\mu}g_{2,s}(x)\right|\leqslant D_{j,\mu}^2\,|A(x)|, \tag{4}$$

where

$$D^2_{j,\mu}=\frac{2}{(r_j^+)!\,(r-\mu-1)!}\left(\frac{2}{x_{j+1}-x_j}\right)^{\mu+1} \times \sum_{i=1}^{\min(r_j^+,\,r-\mu-1)} \binom{r_j^+}{i}[r-\mu-i,i][r,r_j^+-1].$$

In the same way for $x\leqslant(x_j+x_{j+1})/2$ and $s\in[x_j,x_{j+1}]$

$$\left|\frac{\partial^\mu}{\partial s^\mu}g_{1,s}(x)\right|\leqslant D^1_{j,\mu}\,|A(x)|, \tag{5}$$

where

$$D^1_{j,\mu}=\frac{2}{(r_j^-)!\,(r-\mu-1)!}\left(\frac{2}{x_{j+1}-x_j}\right)^{\mu+1} \times \sum_{i=1}^{\min(r_j^-,\,r-\mu-1)} \binom{r_j^-}{i}[r-\mu-i,i][r,r_j^--1].$$

It remains to note that inequality (4) is valid on the set E_j^2 and that

$$\frac{\partial^\mu}{\partial s^\mu}g_{1,s}(x)=\frac{\partial^\mu}{\partial s^\mu}g_{2,s}(x)-\frac{(-1)^\mu(x-s)^{r-\mu-1}}{(r-\mu-1)!} \quad \text{on } E_j^2.$$

Therefore,

$$\left|\frac{\partial^\mu}{\partial s^\mu}g_{1,s}(x)\right|\leqslant D^2_{j,\mu}\,|A(x)|+\frac{|x-s|^{r-\mu-1}}{(r-\mu-1)!} \quad \text{on } E_j^2.$$

Similarly,

$$\left|\frac{\partial^\mu}{\partial s^\mu}g_{2,s}(x)\right|\leqslant D^1_{j,\mu}\,|A(x)|+\frac{|x-s|^{r-\mu-1}}{(r-\mu-1)!} \quad \text{on } E_j^1.$$

It remains to use relation (1) and to denote

$$D_{j,\mu}=\max(D^1_{j,\mu},D^2_{j,\mu}).$$

Remark 1. If $a<x_1$ and $s\in(a,x_1)$ then $G(x,s)=0$ for $x\not\geqslant s$ and $G(x,s)=-(x-s)^{r-1}/(r-1)!$ for $x\not\leqslant s$. Similarly, if $x_n<b$ and $s\in(x_n,b)$ then $G(x,s)=(x-s)^{r-1}/(r-1)!$ for $x\geqslant s$ and $G(x,s)=0$ for $x\leqslant s$.

Remark 2. In what follows we assume that $G(x,s)=0$ when $s<a$ or $s>b$ and that at each point of removable discontinuity the function $G(x,\cdot)$ is defined by continuity. In particular, $G(x_i,\cdot)\equiv 0$ for every i.

5

LEMMA 4. *For fixed* $x \in [a; b]$ *we consider a function* $g_x(s) = G(x, s)$, *where* $G(x, s)$ *is Green's function from Lemma* 3. *We assert that*

(1) $g_x(\cdot)$ *is a piecewise-polynomial function of degree* $\leqslant r-1$ *with the knots at* $x, x_1, x_2, \ldots, x_n, a, b$;

(2) $g_x \in C^{r-\hat{r}-1}(\mathbb{R})$;

(3) *there exists a constant* $D > 0$ *depending only on the position of nodes on* $[a, b]$ *and on their multiplicities, such that for any fixed* x *the inequality* $\operatorname{Var}_{-\infty}^{\infty}[g_x^{(r-\hat{r})}] \leqslant D\,|A_1(x)|$ *holds.*

Proof. The first part of the assertion is already proved and we pass to part 2.

Suppose that $f \in C^r(\mathbb{R})$ and p denotes the Hermitian interpolating polynomial for f with nodes x_i of multiplicities r_i, $a \leqslant x_1 < \cdots < x_n \leqslant b$. Let $y = f - p$. Then

$$y^{(r)}(x) = f^{(r)}(x) \qquad \text{and} \qquad y^{(v)}(x_i) = 0, \qquad v = 0, 1, \ldots, r_i - 1, \quad i = 1, \ldots, n.$$

Therefore,

$$y(x) = \int_{-\infty}^{\infty} G(x, s)\, f^{(r)}(s)\, ds = \int_{-\infty}^{\infty} g_x(s)\, f^{(r)}(s)\, ds.$$

We transform the last expression by performing r times the integration by parts (such a transformation is admissible because of the first assertion of this lemma). This implies that for every $x \notin \{a, b, x_1, \ldots x_n\}$

$$y(x) = \sum_{v=0}^{r-1} (-1)^{v-1} \Bigg[\sigma_{v,x}(a)\, f^{(r-v-1)}(a) + \sigma_{v,x}(b)\, f^{(r-v-1)}(b) + \sigma_{v,x}(x)\, f^{(r-v-1)}(x) + \sum_{i=1}^{n} \sigma_{v,x}(x_i)\, f^{(r-v-1)}(x_i) \Bigg], \tag{6}$$

where $\sigma_{v,x}(t) = g_x^{(v)}(t+0) - g_x^{(v)}(t-0)$.

On the other hand, by the interpolation formula the right-hand part of (6) contains only the terms with $f^{(v)}(x_i)$, $v = 0, 1, \ldots r_i - 1$, $i = 1, \ldots, n$, as is evident from the following argument.

There are functions in $C^r(\mathbb{R})$, even algebraic polynomials, for which all the values $f^{(v)}(a), f^{(v)}(b), f^{(v)}(x), f^{(v)}(x_i)$ $(v \leqslant r-1, i \leqslant n)$ vanish except one of them (arbitrarily taken) equaling 1. Therefore, $\sigma_{v,x}(a) = \sigma_{v,x}(b) = 0$, $v \leqslant r-1$.[1] $\sigma_{v,x}(x) = 0$ for $v \leqslant r-2$ and $\sigma_{r-1,x}(x) = (-1)^r$. Finally,

[1] Thus, even if $x_1 \neq a$ or $x_2 \neq b$ the memebers in (6) depending on a or b may be omitted.

$\sigma_{v,x}(x_i)=0$ for $v \leqslant r-r_i-1$, $i=1, 2, ..., n$. This implies the continuity of $g_x^{(v)}(\cdot)$ for $v=0, 1, ..., r-\hat{r}-1$.

Now we pass to the proof of the last assertion. Taking into account that $\sigma_{r-\hat{r},x}(a)=\sigma_{r-\hat{r},x}(b)=0$ we can write

$$\mathrm{Var}_{-\infty}^{\infty}\,[g_x^{(r-\hat{r})}]=|\sigma_{r-\hat{r},x}(x)|+\sum_{i=1}^{n}|\sigma_{r-\hat{r},x}(x_i)|+\int_a^b |g_x^{(r-\hat{r}+1)}(s)|\,ds. \quad (7)$$

If $\hat{r}=1$ (i.e., in the case of Lagrange interpolation), then $r-\hat{r}+1=r$ and consequently the last integral equals 0. Comparison of the equality (6) with the Lagrange interpolation formula shows that in this case $\sigma_{r-\hat{r},x}(x)=(-1)^r$, $\sigma_{r-\hat{r},x}(x_i)=(-1)^{r-1}\,l_i(x)$, where l_i is the ith fundamental polynomial for the nodes $x_1, ..., x_n$ of multiplicity 1. Therefore, for $\hat{r}=1$

$$\mathrm{Var}_{-\infty}^{\infty}\,[g_x^{(r-\hat{r})}]\leqslant D\equiv D\,|A_1(x)|,$$

where is $D=1+\sum_{i=1}^{n}\|l_i\|$.

Now let $\hat{r}\geqslant 2$. In this case $\sigma_{r-\hat{r},x}(x)=0$. By Lemma 3 the right-hand sum in (7) is not greater than $D_1\,|A(x)|+\min_j(|x-x_j|^{\hat{r}-1}/(\hat{r}-1)!)$, where $D_1=D_{1,r-\hat{r}}+2\sum_{j=2}^{n-1}D_{j,r-\hat{r}}+D_{n,r-\hat{r}}$, and the integral in (7) by the same lemma is not greater than $D_2\,|A(x)|+\min_j(|x-x_j|^{\hat{r}-1}/(\hat{r}-1)!$, where $D_2=2\sum_{j=2}^{n-1}(x_{j+1}-x_j)\,D_{j,r-\hat{r}+1}$. Hence,

$$\mathrm{Var}_{-\infty}^{\infty}\,[g_x^{(r-\hat{r})}]\leqslant(D_1+D_2)\,|A(x)|+\frac{2}{(\hat{r}-1)!}\min|x-x_j|^{\hat{r}-1}. \quad (8)$$

Since $|A(x)|=|\alpha(x)|\,|A_1(x)|\leqslant\alpha\,|A_1(x)|$, it remains to estimate the quotient $\psi(x)=\min_j|x-x_j|^{\hat{r}-1}/|A_1(x)|$.

We have $\psi(x)=(|x-x_i|^{\hat{r}-1-r_i'}/\prod_{k\neq i}|x-x_k|^{r_k'})$ on segment Δ_i, where $\Delta_1=[a; (x_1+x_2)/2]$, $\Delta_i=[(x_{i-1}+x_i)/2; (x_i+x_{i+1})/2]$ for $i=2, ..., n-1$ and $\Delta_n=[(x_{n-1}+x_n)/2; b]$. It is easy to see that function $\psi(x)$ is bounded on $[a, b]$, the number $\psi=\max_{a\leqslant x\leqslant b}\psi(x)$ depends only on the multiplicities of nodes x_i and on their position on $[a; b]$ and that

$$\min_j|x-x_j|^{\hat{r}-1}\leqslant\psi\,|A_1(x)|. \quad (9)$$

Now from (8) and (9) follows the inequality

$$\mathrm{Var}_{-\infty}^{\infty}\,[g_x^{(r-\hat{r})}]\leqslant D\,|A_1(x)|,$$

where $D=(D_1+D_2)\alpha+2\psi/(\hat{r}-1)!$.

The proof of Lemma 4 is complete.

6

LEMMA 5. *In the notation of the theorem for every* $x \in [a; b]$ *the following inequalities are valid*:

$$|f(x)-p(x)| \leqslant \frac{1}{r!} \|f^{(r)}\| \, |A(x)| \qquad \text{if} \quad f \in C^r[a; b], \tag{10}$$

$$|f(x)-p(x)| \leqslant D \, \|f^{(\hat{r}-1)}\| \, |A_1(x)| \qquad \text{if} \quad f \in C^{\hat{r}-1}[a; b]. \tag{11}$$

(*D is the constant from the Lemma* 4).

Proof. By the properties of divided differences we have first,

$$f(x)-p(x)=[f; x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n] \, A(x),$$
$$r_1 \text{ times}, \ldots, r_n \text{ times}$$

and secondly, if $f \in C^r[a; b]$, then there exists $\xi \in [a; b]$ such that

$$f(x)-p(x)=\frac{1}{r!} f^{(r)}(\xi) \, A(x).$$

This implies the inequality (10).

Passing to the second assertion of the Lemma 5 we prove for any $f \in C^{\hat{r}-1}[a; b]$ that

$$f(x)-p(x)=(-1)^{r-\hat{r}+1} \int_a^b f^{(\hat{r}-1)}(s) \, d[g_x^{(r-\hat{r})}(s)]. \tag{12}$$

It is easy to see that it suffices to obtain the representation (12) for any function from $C^r[a; b]$. But for such a function

$$f(x)-p(x)=\int_a^b f^{(r)}(s) \, g_x(s) \, ds=(-1)^{r-\hat{r}} \int_a^b f^{(\hat{r})}(s) \, g_x^{(r-\hat{r})}(s) \, ds$$
$$=(-1)^{r-\hat{r}+1} \int_a^b f^{(\hat{r}-1)}(s) \, d[g_x^{(r-\hat{r})}(s)]$$

(when integrating by parts we make use of the properties of $g_x(\cdot)$ from Lemma 4).

Now from the representation (12) it follows by Lemma 4 that for any $f \in C^{\hat{r}-1}[a; b]$

$$|f(x)-p(x)| \leqslant \|f^{(\hat{r}-1)}\| \operatorname{Var}_{-\infty}^{\infty} [g_x^{(r-\hat{r})}] \leqslant D \, \|f^{(\hat{r}-1)}\| \, |A_1(x)|.$$

Lemma 5 is proved.

In order to obtain the main theorem we use here, just as in our previous paper [3], one more auxiliary result.

LEMMA 6. *For $l \in \mathbb{N}_0$, $k \in \mathbb{N}$, and $m = l + k$ let us consider all possible representations of $f \in C^l[a; b]$ in the form $f = f_0 + f_1$, where $f_0 \in C^l[a; b]$, $f_1 \in C^m[a; b]$. Then for any positive numbers A_0, A, h $(h \leqslant (b-a)/k)$ the following inequality holds*

$$\inf_{f = f_0 + f_1} \{A_0 \|f_0^{(l)}\| + A_1 h^k \|f_1^{(m)}\|\} \leqslant \{A_0 + A_1 B_k) \, \omega_k(f^{(l)}; h),$$

where B_k depends only on k.

This lemma may be considered as known (see, for instance, the work of Yu. A. Brudnyi [4]). For the proof one may also consult the above-mentioned paper [3] (see the proof of the Lemma 5).

7

Proof of the Theorem. Let us consider a linear operator L on $C^{r-1}[a; b]$, such that (in notations of the Theorem) $\mathrm{Lf} = y = f - p$. If we represent f arbitrarily in the form $f = f_0 + f_1$, where $f_0 \in C^{\hat{r}-1}[a; b]$, $f_1 \in C^r[a; b]$ than by applying Lemma 5 we obtain for $x \in [a; b]$

$$|Lf(x)| \leqslant |Lf_0(x)| + |Lf_1(x)| < D \, \|f_0^{(\hat{r}-1)}\| \, |A_1(x)| + \frac{\|f_1^{(r)}}{r!} |A(x)|.$$

Therefore,

$$|y(x)| \leqslant |A_1(x)| \inf_{f = f_0 + f_1} \left\{ D \, \|f_0^{(\hat{r}-1)}\| + \frac{1}{r!} |\alpha(x)| \, \|f_1^{(r)}\| \right\}$$

$$= |A_1(x)| \inf_{f = f_0 + f_1} \left\{ D \, \|f_0^{(\hat{r}-1)}\| + \frac{\alpha}{r!} \left(\frac{r - \hat{r} + 1}{b - a} h(x) \right)^{r - \hat{r} + 1} \|f_1^{(r)}\| \right\},$$

where $h(x) = ((b-a)/(r - \hat{r} + 1))(|\alpha(x)|/\alpha)^{1/(r - \hat{r} + 1)}$ (we remind the reader that $\alpha = \max_{a \leqslant x \leqslant b} |\alpha(x)|$ and thus $0 < h(x) \leqslant ((b-a)/(r - \hat{r} + 1)))$. Then Lemma 6 implies

$$|y(x)| \leqslant C \, |A_1(x)| \, \omega_{r - \hat{r} + 1}(f^{(\hat{r}-1)}; h(x)),$$

where the constant $C = D + (\alpha/r!) \, B_{r - \hat{r} + 1}((r - \hat{r} + 1)/(b-a))^{r - \hat{r} + 1}$ depends only on the nodes of interpolation and on their multiplicities.

The theorem is now proved.

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